Cambridge International Examinations
Cambridge International Advanced Level

FURTHER MATHEMATICS **9231/22**

Paper 2 **October/November 2016**

MARK SCHEME

Maximum Mark: 100

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge will not enter into discussions about these mark schemes.

Cambridge is publishing the mark schemes for the October/November 2016 series for most Cambridge IGCSE®, Cambridge International A and AS Level components and some Cambridge O Level components.

Mark Scheme Notes

Marks are of the following three types:

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

- When a part of a question has two or more "method" steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

- The symbol √ implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously "correct" answers or results obtained from incorrect working.

  - Note: B2 or A2 means that the candidate can earn 2 or 0.
    B2/1/0 means that the candidate can earn anything from 0 to 2.

  The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.

- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking g equal to 9.8 or 9.81 instead of 10.

The following abbreviations may be used in a mark scheme or used on the scripts:

AEF/OE Any Equivalent Form (of answer is equally acceptable) / Or Equivalent

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

CAO Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)

CWO Correct Working Only – often written by a 'fortuitous' answer

ISW Ignore Subsequent Working

SOI Seen or implied

SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Penalties

MR –1 A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become "follow through √ " marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR –2 penalty may be applied in particular cases if agreed at the coordination meeting.

PA –1 This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

1 (i)	Find ω^2 from $v^2 = \omega^2 (a^2 - x^2)$, or ω: Find max. acceln. from $d^2x/dt^2 = -\omega^2 x$, √ on ω^2:	$4^2 = \omega^2 (3^2 - 1^2)$, $\omega^2 = 2$ $2 \times 3 = 6$ [m s^{-2}] (allow –6)	**M1 A1** **A1**√	[3]
(ii)	Find no. of oscillations in 60 s from $T = 2\pi/\omega$: and hence no. of complete oscillations (allow M1 A0 for $60 / (\pi/\omega)$ [= 27])	$60 / (2\pi/\omega)$ [= 60 / 4·443 = 13·5] 13	**M1** **A1**	[2]
(iii)	Find time from *A* to *C*, e.g.:	$\omega^{-1} \sin^{-1} (1) + \omega^{-1} \sin^{-1} ⅓$ *or* $¼T + \omega^{-1} \sin^{-1} ⅓$ [= 1·111 + 0·240] *or* $\omega^{-1} \cos^{-1} (-⅓)$ *or* $½T - \omega^{-1} \cos^{-1} ⅓$ [= 2·221 – 0·870] = 1·91 / ω ; = 1·35 [s]	**M1** **A1; A1**	[3]
2 (i)	*EITHER:* Find comps. of speed after colln. at *E*: Relate *v* to *u*, or v^2 to u^2: *OR:* Relate angle β after colln. to *u*, *v*: Find tan β, or β: Eliminate β from either eqn. above, e.g.:	$v \cos 45°$ // to wall *and* $¾\, v \sin 45°$ ⊥ to wall $\sqrt{\{(v / \sqrt{2})^2 + (¾\, v / \sqrt{2})^2\}} = ¼\, u$ $(5/4\sqrt{2})\, v = ¼\, u$ $v = (\sqrt{2}/5)\, u$ A.G $¼\, u \cos \beta = v \cos 45°$ *and* $¼\, u \sin \beta = ¾\, v \sin 45°$ $\tan \beta = ¾$ *or* $\beta = 36·9°$ $¼\, u \times (4/5) = v / \sqrt{2}$ $v = (\sqrt{2}/5)\, u$ A.G.	 **M1 A1** **M1** **A1** **A1** **(M1 A1)** **(A1)** **(M1)** **(A1)**	[5]
(ii)	Relate comps. of speed // to wall after colln. at *D*: Find cos α: Find α: Relate comps. of speed ⊥ to wall after colln. at *D*: Find *e*:	$v \cos 75° = u \cos \alpha$ $\cos \alpha = (\sqrt{2}/5) \cos 75°$ [= 0·0732] $\alpha = 85·8°$ *or* 1·50 rads $v \sin 75° = eu \sin \alpha$ $e = (\sqrt{2}/5) \sin 75° / \sin \alpha$ *or* = $\tan 75° / \tan \alpha = 0·274$	**M1** **A1** **A1** **M1** **A1**	[5]

3 (i)	Verify $\sin\theta$ from triangle CDE where E is level with C and vertically below D:	$\sin\theta = 6a / 10a = 3/5$ A.G.	B1	[1]
(ii)	Resolve forces on object vertically: (may be needed in part (iii) only)	$R_A = (k + 2)W$	M1 A1	
	Take moments about B :	$F_A\,7a + Wa + kWa(1 + 5\cos\theta)$ $+ (W - R_A)\,a(1 + 10\cos\theta) = 0$		
	or D:	$F_A\,7a + kW\,5a\cos\theta$ $+ (W - R_A)\,10a\cos\theta = 0$		
	or A :	$R_B\,7a = kW\,5a\cos\theta + W\,10a\cos\theta$		
	or C :	$F_A\,a + R_B\,6a = W\,10a\cos\theta + kW\,5a\cos\theta$		
	or centre of rod:	$F_A(5a\sin\theta + a) + R_B\,5a\sin\theta + W\,5a\cos\theta =$ $R_A\,5a\cos\theta + W\,5a\cos\theta$	M1 A1	
	Find F_A using $R_B = F_A$, $\sin\theta = 3/5$, $\cos\theta = 4/5$ as necessary: B:	$7F_A = 9R_A - 5kW - 10W = 4kW + 8W$		
	D:	$7F_A = 8R_A - 4kW - 8W = 4kW + 8W$		
	A *or* C:	$7F_A = 4kW + 8W$		
	Centre of rod:	$7F_A = 4R_A$ [so R_A is not reqd. here]	A1	
	Find μ :	$\mu = F_A / R_A = 4/7$ *or* 0·	M1 A1	[7]
(iii)	Equate resultant force at A to $W\sqrt{(65)}$:	$(k + 2)^2 + (4/7)^2 (k + 2)^2 = 65$	M1 A1	
	Solve to verify value of k:	$k^2 + 4k - 45 = (k - 5)(k + 9) = 0$ *or* $(k + 2)^2 = 49$ so $k = 5$ A.G.	A1	[3]

4 (i)	Find v_1^2 at lowest point from consvn. of energy: Verify new v_2 from consvn. of momentum:	$\frac{1}{2}mv_1^2 = \frac{1}{2}mu^2 + 2mga$ $\frac{1}{2}mv_1^2 = \frac{1}{2}mu^2 + 2mga$ $Mv_2 = mv_1$ with $M = (\lambda + 1)\, m$ $v_2 = v_1 / (\lambda + 1)$ $= \{5/(\lambda + 1)\}\sqrt{(\frac{1}{3} ag)}$ A.G.	**M1 A1** **M1 A1**	[4]
(ii)	Use $F = ma$ radially at slack point with $T = 0$, e.g.: (θ is angle of string with upward vertical) Find v_3^2 at slack point from consvn of energy: **SR**: Lose max 1 mark if mass is m in either eqn. Eliminate v_3^2 [$= ag/3$] using $\cos\theta = \frac{1}{3}$: Substitute for v_2 to find λ:	$Mv_3^2/a = Mg\cos\theta$ $\frac{1}{2} Mv_3^2 = \frac{1}{2} Mv_2^2 - Mg\,(4a/3)$ $ag/3 = v_2^2 - 8ag/3$ $3ag = \{25/(\lambda + 1)^2\}\, ag/3$ $(\lambda + 1)^2 = 25/9,\ \lambda = \frac{2}{3}$ [rejecting –8/3]	**B1** **M1 A1** **M1** **M1 A1**	[6]
(iii)	Use $F = ma$ radially just before collision: Use $F = ma$ radially just after collision: Find change in tension (either sign, AEF):	$T_1 = mv_1^2/a + mg$ [$= (25/3 + 1)mg = 28mg/3$] $T_2 = Mv_2^2/a + Mg$ [$= (3 + 1)(5m/3)g = 20mg/3$] $(25mg/3)\{\lambda/(\lambda + 1)\} - \lambda mg$ $= 8mg/3$ *or* $2{\cdot}67mg$	 **B1** **B1** **A1** **A1**	[4]
5 (i)	Find a from mean:	$a = 1/10\,000$ *or* 10^{-4}	**B1**	[1]
(ii)	Find P(X < 15 000):	$1 - e^{-15\,000a} = 1 - e^{-1.5} = 0{\cdot}777$	**M1 A1**	[2]
(iii)	Formulate condition for d: (M0 for $1 - e^{-ad} = 0{\cdot}75$, giving $d = 13\,900$)) Rearrange and take logs to give d:	$1 - (1 - e^{-ad}) = 0{\cdot}75$ $d = -(\ln 0{\cdot}75) / a$ $= 2877$ or 2880	**M1** **A1** **A1**	[3]

6	Estimate population variance: (allow biased here: 0·098 *or* $0{\cdot}3130^2$) State hypotheses (AEF; B0 for $\bar{x}$): Calculate value of t (either sign; to 3 s.f.): State or use correct tabular t-value (to 3 s.f.): (or can compare $\bar{x}$ with 6·4 – 0·224 = 6·176) State or imply valid method for conclusion e.g.: Conclusion (AEF, requires both values correct):	$s^2 = 0{\cdot}784 / 7$ $= 0{\cdot}112$ *or* 14/125 *or* $0{\cdot}3347^2$ H_0: $\mu = 6{\cdot}44$, H_1: $\mu < 6{\cdot}44$ $t = (6{\cdot}44 - \bar{x})/(s/\sqrt{8}) = 1{\cdot}64$ $t_{7,\,0.95} = 1{\cdot}89[5]$ Accept H_0 if $t <$ tabular value 1·64 < 1·89 so popln. mean not less than 6·44	 **M1** **B1** **M1 A1** **B1** **M1** **A1**	[7]
7 (i)	Find or state distribution function F(x) for $2 \leqslant x \leqslant 4$: Use F(2) = 0 or F(4) = 1 to find F(x):	$F(x) = \int f(x)\,dx = x^2/12 + c$ $F(x) = x^2/12 - ⅓\ [(2 \leqslant x \leqslant 4)]$, $0\ (x < 2),\ 1\ (x > 4)$	 **M1 A1** **A1**	[3]
(ii)	Find or state G(y) from $Y = X^3$ for $2 \leqslant x \leqslant 4$: (allow < or ⩽ throughout) (A0 if G(y) incorrect) Find g(y) by differentiation:	$G(y) = P(Y < y) = P(X^3 < y)$ $= P(X < y^{1/3}) = F(y^{1/3})$ $= y^{2/3}/12 - ⅓$ $g(y) = (1 / 18)\, y^{-1/3}$ [for $8 \leqslant y \leqslant 64$, 0 otherwise]	 **M1 A1**	[2]
(iii)	Formulate condition for k: (M0 for 7/12 = G(k)) Find k:	(AEF) $7/12 = 1 - G(k) = 1 - k^{2/3}/12 + ⅓$ $k^{2/3} = 9$, $k = 27$	**M1 A1** **A1**	[3]

8 **(i)**	*EITHER:* Estimate *P*'s popln. variance (to 3 d.p.): (allow biased here: 400) Estimate *Q*'s popln. variance (to 3 d.p.): (allow biased here: 314) Find pooled estimate of common variance: *OR:* Find pooled estimate of common variance:	$s_P^2 = (1\,560\,000 - 9600^2/60) / 59$ [= 406·78] $s_Q^2 = (1\,052\,500 - 7200^2/50) / 49$ [= 320·41] $s^2 = (59\, s_P^2 + 49\, s_Q^2) / 108$ = 367·6 *or* 368 *or* 9925/27$s^2 = (1\,560\,000 - 9600^2/60$ $s^2 = (1\,560\,000 - 9600^2/60$ $+ 1\,052\,500 - 7200^2/50) / 108$ = 367·6 *or* 368 *or* 9925/27	**M1** **M1** **M1 A1** **(M3 A1)**	[4]
(ii)	Find confidence interval for the difference: Use appropriate tabular value (to 2 d.p.): Evaluate confidence interval (AEF, to 1 d.p.): **SR** Using combined variance $s_P^2/60 + s_Q^2/50 = 13{\cdot}19$: (i) M1, M1 as above; then M0 A0 (max 2/4) (ii) M1 A0 for 9600/60 – 7200/50 ± *z s* A1 for tabular value as above M1 A0 for evaluating interval 16 ± 7·1 (*or* 7·2) (max 3/5)	9600/60 – 7200/50 [= 160 – 144] $\pm z\, s\sqrt{(60^{-1} + 50^{-1})}$ $z_{0.975} = 1{\cdot}96$ *or* $t_{120,\, 0.975} = 1{\cdot}98$ 16 ± 7·2 *or* [8·8, 23·2] *or* 16 ± 7·3 *or* [8·7, 23·3]	 **M1 A1** **A1** **M1 A1**	[5]

9 (i)	Find mean and variance of sample data.: State valid reason why Poisson distn. suitable (AEF): (allow unsuitable since $4 \neq 3\cdot57$)	$\bar{x} = 240/60 = 4$ $\sigma^2 = 1174/60 - 4^2 = 3\cdot57$ $4 \approx 3\cdot57$ (no √ on $\bar{x}, \sigma^2$)	**B1** **B1** **B1**	[3]
(ii)	Find expected values $60\lambda^r e^{-\lambda} / r!$ with $\lambda = 4$:	$4\cdot40$; $11\cdot7_{[2]}$ (to 1 d.p.)	**B1; B1**	[2]
(iii)	State (at least) null hypothesis (AEF): Combine cells so that all exp. value ⩾ 5: Calculate value of χ^2 (to 2 d.p.; A1 dep *M1): State or use consistent tabular value (to 2 d.p.): [or fewer or no cells combined: State or imply valid method for conclusion e.g.: Conclusion (AEF, requires both values correct):	H_0: [Poisson] distribution fits data O_i : 4 12 … 6 8 E_i : <u>5·50</u> 8·79 … 6·25 <u>6·64</u> $\chi^2 = 0\cdot409 + 1\cdot172 + 1\cdot181$ $+ 0\cdot044 + 1\cdot397 + 0\cdot81 + 0\cdot279$ $= 5\cdot29$ 7 cells: $\chi_{5,\,0.9}{}^2 = 9\cdot236$ 8 cells: $\chi_{6,\,0.9}{}^2 = 10\cdot64$ 9 cells: $\chi_{7,\,0.9}{}^2 = 12\cdot02$ 10 cells: $\chi_{8,\,0.9}{}^2 = 13\cdot36$] Accept H_0 if χ^2 < tabular value $5\cdot29 < 9\cdot24$ so distn. fits	**B1** ***M1 A1** **M1 DA1** **B1√** **M1** **A1**	[8]

10 (a) (i)	Verify MI of either disc about axis l at O:	$I_{disc} = ½\,(¼ma^2)$ $+ ½m\,\{(3a/2)^2 + (2a)^2\}$ $= ma^2/8 + 25ma^2/8$ $= 13ma^2 / 4$ A.G.	**M1 A1** **A1**	[3]
(ii)	Find or state MI of rod OC about l: Find MI of rod AB about l: Verify MI of object about l:	$I_{OC} = (4/3)\,ma^2$ $I_{AB} = ⅓\,2m\,(3a/2)^2 + 2m\,(2a)^2$ $= (19/2)\,ma^2$ $I = (2 \times 13/4 + 4/3 + 19/2)\,ma^2$ $= (52/3)\,ma^2$ A.G.	**B1** **M1 A1** **A1**	[4]
(iii)	Find and use initial angular speed: Find initial rotational KE: Find gain in P.E. at instantaneous rest: Verify cos θ by equating KE and PE:	$\omega_0 = \sqrt{(2ag)} / 2a$ *or* $\sqrt{(g / 2a)}$ $½\,I\,\omega_0^2 = ½\,(52/3)\,ma^2 \times (g/2a)$ $= (13/3)\,mga$ $(3mg \times 2a + mga)\,(1 – \cos\theta)$ *or* $(4mg \times 7a/4)\,(1 – \cos\theta)$ $= 7mga\,(1 – \cos\theta)$ $1 – \cos\theta = (13/3)\,mga / 7mga$ $= 13/21,\ \cos\theta = 8/21$ A.G.	**B1** **M1 A1** **M1 A1** **M1 A1**	[7]

(b) (i)	*EITHER:* Find $\bar{x}$, $\bar{y}$ by solving simultaneous eqns: Hence find missing values x_5, y_5: *OR:* Formulate simultaneous eqns for x_5, y_5: (AEF, M1 for either) Hence find missing values x_5, y_5:	$\bar{x} = 5$; $\bar{y} = 6$ $22 + x_5 = 5 \times 5$, $x_5 = 3$ $25 + y_5 = 5 \times 6$, $y_5 = 5$ $25 + y_5 = 5 \times 4{\cdot}5 + 0{\cdot}3\ (22 + x_5)$ $22 + x_5 = 3\ (25 + y_5) - 5 \times 13$ $y_5 = 0{\cdot}3\ x_5 + 4{\cdot}1$, $x_5 = 3\ y_5 - 12$ $x_5 = 3$; $y_5 = 5$	**M1, A1; A1** **A1** **A1** **(M1 A1)** **(A1)** **(A1 A1)**	[5]
(ii)	Find correlation coefficient r: (A0 for – 0·949 or ± 0·949)	$r = \sqrt{(0{\cdot}3 \times 3)}$ *or* $12/\sqrt{(40 \times 4)} = 0{\cdot}949$	**M1 A1**	[2]
(iii)	Find corresponding summations for combined data: (B1 needs all 5 correct) Find correlation coefficient r': (M0 if based on *B* only) State both hypotheses (B0 for r …): State or use correct tabular two-tail r-value: State or imply valid method for conclusion e.g.: Correct conclusion (AEF, dep *A1, *B1):	$\sum x = 25 + 20 = 45$ $\sum x^2 = 165 + 100 = 265$ $\sum y = 30 + 17 = 47$ $\sum y^2 = 184 + 69 = 253$ $\sum xy = 162 + 75 = 237$ $S_{xy} = 237 - 45 \times 47/10 = 25{\cdot}5$ $S_{xy} = 265 - 45^2/10 = 62{\cdot}5$ $S_{yy} = 253 - 47^2/10 = 32{\cdot}1$ $r' = S_{xy} / \sqrt{(S_{xx} S_{yy})}$ *or* $\sqrt{\{(S_{xy} / S_{xy})(S_{xy} / S_{yy})\}}$ $= 25{\cdot}5 / 44{\cdot}79$ *or* $\sqrt{(0{\cdot}408 \times 0{\cdot}7944)}$ $= 0{\cdot}569$ H_0: $\rho = 0$, H_1: $\rho \neq 0$ $r_{10,\ 5\%} = 0{\cdot}632$ Accept H_0 if $\lvert r' \rvert <$ tab. value (AEF) Popln. pmcc not different from 0	 **B1** **M1 *A1** **B1** ***B1** **M1** **DA1**	[7]